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T-STRESS IN PIEZOELECTRIC SOLID *

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Abstract: *The non-singular and bounded terms for stresses near the crack tip were investigated. The crack problem in a transversely isotropic piezoelectric solid for the plane problem was dealt with. The principle of superposition and the Plemelj formulation were introduced. The non-singular terms are given by solving Rieman-Hilbert problem. It is shown that the non-singular terms are influenced by the elastic and electric constants.*

Key words: piezoelectric solid; crack; non-singular term

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Introduction

The elastic non-singular terms (NST) can be identified with the stresses parallel to the crack[1]. It is well-known that these terms can play an important role in fracture analysis and their effects should understand. The work by Lavsson and Carlsson[2] indicated that the sign and magnitude of the NST substantially alter the size and shape of the plain-strain crack-tip plastic zone at finite load levels. Bilby *et al*.[3] showed that the NST extends the range of validity of the small-scale yielding conditions at infinite strains. Betegon and Hancock[4] characterized the elastic-plastic crack tip field by using a two-parameter law, i.e., *J*-integral and *T*-stress. Coteerell and Rice[5] and Fleck *et al*.[6] showed that the NST governs the stability of straight crack path under model Ⅰ loading conditions. Zhao Li-guo and Chen Yi-heng[7] studied the effect of *T*-stress in microcrack shielding problem by solving the interaction problem of a macrocrack with near tip microcracks applying a discrete model. Sham[8] calculated the NST in isotropic elastic solids by using higher order weigh functions. Ma *et al*.[9] obtained the NST for multiple cracks in anisotropic elastic solids by using the principle of superposition. However, the calculation of the NST in piezoelectric solid has not been attempted. The present paper calculates

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the NST at the crack tip in transversely isotropic piezoelectric solid for the plane problem (Fig. 1). The poling direction x_2 is perpendicular to the crack front. Solution to a piezoelectric crack problem, in which a finite crack of length $2a$ is embeded in an unbounded piezoelectric medium subject to far uniform mechanical and electrical loads, can be obtained by superposing two solutions as follows:

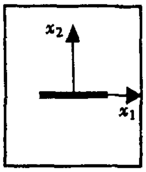

Fig. 1 Piezoelectric material with a crack

$$T = T^{\infty} + T^{s}. \tag{1}$$

The superscript ∞ refers to the uniform solution for piezoelectric medium without the crack, and the superscript s refers to the uniform solution for the same body with the opposite far uniform mechanical and electrical loads acting on the crack faces. The second solution is this paper's purpose and the results should serve a useful purpose for fracture analysis.

1 Basic Approach

For the plane strain problem, the stress components and electric displacements can be expressed as follows[10]:

$$\begin{cases} \sigma_{11} = 2\mathrm{Re}\sum_{k=1}^{3}\mu_k^2\Phi_k(z_k),\ \sigma_{22} = 2\mathrm{Re}\sum_{k=1}^{3}\Phi_k(z_k),\ \sigma_{12} = -2\mathrm{Re}\sum_{k=1}^{3}\mu_k\Phi_k(z_k), \\ D_1 = 2\mathrm{Re}\sum_{k=1}^{3}\lambda_k\mu_k\Phi_k(z_k),\ D_2 = -2\mathrm{Re}\sum_{k=1}^{3}\lambda_k\Phi_k(z_k). \end{cases} \tag{2a ~ e}$$

The functions $\Phi_k(z_k)$ are the complex potentials; $z_k = x_1 + \mu_2 x_2$; μ_k and their conjugates are roots of the characteristic equation; λ_k is defined in Ref. [10].

In order to calculate the NST, the following three problems, A, B and C are considered:

Problem A The crack problem with plane symmetric tractions $-\sigma_{22}^{\infty}(x_1)$ applied to the crack.

Problem B The crack problem with plane skew-symmetric tractions $-\sigma_{22}^{\infty}(x_1)$ applied to the crack.

Problem C The crack problem with plane symmetric electric displacement $-D_2^{\infty}$ to the crack.

2 Problem A

On the real axis x_1, $\sigma_{12}^{\infty} = 0$, $D_2^{\infty} = 0$ and $z_1 = z_2 = z_3 = t$, then from Eq. (2c) and Eq. (2e) yields

$$\sum_{k=1}^{3}\mu_k\Phi_k(t) = 0,\quad \sum_{k=1}^{3}\lambda_k\Phi_k(t) = 0 \quad \text{for all } t. \tag{3}$$

The boundary values on the upper and lower side are denoted by the plus (+) and minus (−), respectively. Eq. (2b) yields

$$\begin{cases} \sigma_{22}^{+} = \dfrac{S}{P_k}\Phi_k^{+}(t) + \dfrac{S}{\bar{P}_k}\bar{\Phi}_k^{-}(t), \\ \sigma_{22}^{-} = \dfrac{S}{P_k}\Phi_k^{-}(t) + \dfrac{\bar{S}}{\bar{P}_k}\bar{\Phi}_k^{+}(t) \qquad \text{for } |t| < a;\ k = 1, 2, 3, \end{cases} \tag{4}$$

where the repeated subscript does not denote the summation, and

$$\begin{cases} S = \mu_1(\lambda_2 - \lambda_3) + \mu_2(\lambda_3 - \lambda_1) + \mu_3(\lambda_1 - \lambda_2), \\ P_1 = \mu_2\lambda_3 - \mu_3\lambda_2;\ P_2 = \mu_3\lambda_1 - \mu_1\lambda_3;\ P_3 = \mu_1\lambda_2 - \mu_2\lambda_1. \end{cases} \tag{5}$$

This leads to Rieman-Hilbert problem:

$$\begin{cases} \left[\frac{S}{P_k}\Phi_k(t) + \frac{\bar{S}}{\bar{P}_k}\bar{\Phi}_k(t)\right]^+ + \left[\frac{S}{P_k}\Phi_k(t) + \frac{\bar{S}}{\bar{P}_k}\bar{\Phi}_k(t)\right]^- = -2\sigma_{22}^\infty(t) & \text{for } |t| < a, \\ \left[\frac{S}{P_k}\Phi_k(t) - \frac{\bar{S}}{\bar{P}_k}\bar{\Phi}_k(t)\right]^+ + \left[\frac{S}{P_k}\Phi_k(t) - \frac{\bar{S}}{\bar{P}_k}\bar{\Phi}_k(t)\right]^- = 0 & \text{for } |t| < a. \end{cases} \tag{6}$$

The solution can be expressed by

$$\Phi_k(z_k) = \frac{P_k}{S}\frac{-1}{2\pi \mathrm{i} X(z_k)}\int_{-a}^{a}\frac{X^+(t)\sigma_{22}^\infty(t)}{t - z_k}\mathrm{d}t \qquad (k = 1, 2, 3), \tag{7}$$

where $X(z_j) = (z_j^2 - a^2)^{1/2}$, $j = 1, 2, 3$.

Using the Plemelj formulation in Eq.(7) and let $z_k \to t^+$, we obtain

$$\Phi_k^+(t) = \frac{P_k}{S}\left[\frac{G(t)}{2} - \frac{\sigma_{22}^\infty(t)}{2}\right] \qquad \text{for } |t| < a, \tag{8}$$

in which $G(t) = \frac{-1}{\pi \mathrm{i} X(t)}\int_{-a}^{a}\frac{X(s)\sigma_{22}^\infty(s)}{s - t}\mathrm{d}s$.

On the other hand, from Eq.(2a)

$$\sigma_{11}^+(t) = \mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 P_k}{S}G(t) - \sigma_{22}^\infty(t)\mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 P_k}{S} \qquad \text{for } |t| < a. \tag{9}$$

Letting $t \to a$, the NST is given by

$$T = -\sigma_{22}^\infty(a)\mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 P_k}{S}. \tag{10}$$

3 Problem B

On the real axis x_1, $\sigma_{22}^\infty = 0$, $D_2^\infty = 0$ and $z_1 = z_2 = z_3 = t$, then from Eq.(2c) and Eq.(2e) yields

$$\sum_{k=1}^{3}\Phi_k(k) = 0, \quad \sum_{k=1}^{3}\lambda_k\Phi_k(t) = 0 \qquad \text{for all } t. \tag{11}$$

The boundary values on σ_{12} in Eq.(2c) become

$$\begin{cases} \sigma_{12}^+ = \frac{S}{Q_k}\Phi_k^+(t) + \frac{\bar{S}}{\bar{Q}_k}\bar{\Phi}_k^-(t), \\ \sigma_{12}^- = \frac{S}{Q_k}\Phi_k^-(t) + \frac{\bar{S}}{\bar{Q}_k}\bar{\Phi}_k^+(t), & \text{for } |t| < a,\ k = 1, 2, 3, \end{cases} \tag{12}$$

in which $Q_1 = \lambda_2 - \lambda_3$; $Q_2 = \lambda_3 - \lambda_1$ and $Q_3 = \lambda_1 - \lambda_2$.

The Rieman-Hilbert formulation is obtained

$$\begin{cases} \left[\frac{S}{Q_k}\Phi_k(t) + \frac{\bar{S}}{\bar{Q}_k}\bar{\Phi}_k(t)\right]^+ + \left[\frac{S}{Q_k}\Phi_k(t) + \frac{\bar{S}}{\bar{Q}_k}\bar{\Phi}_k(t)\right]^- = -2\sigma_{12}^\infty & \text{for } |t| < a, \\ \left[\frac{S}{Q_k}\Phi_k(t) - \frac{\bar{S}}{\bar{Q}_k}\bar{\Phi}_k(t)\right]^+ + \left[\frac{S}{Q_k}\Phi_k(t) - \frac{\bar{S}}{\bar{Q}_k}\bar{\Phi}_k(t)\right]^- = 0 & \text{for } |t| < a. \end{cases} \tag{13}$$

The solution can be expressed by

$$\Phi_k(z_k) = \frac{Q_k}{S}\frac{-1}{2\pi \mathrm{i} X(z_k)}\int_{-a}^{a}\frac{X^+(t)\sigma_{12}^\infty(t)}{t - z_k}\mathrm{d}t \qquad (k = 1, 2, 3). \tag{14}$$

Using the Plemelj formulation in Eq.(14) and let $z_k \to t^+$, we obtain

$$\Phi_k^+(t) = \frac{Q_k}{S}\left[\frac{H(t)}{2} - \frac{\sigma_{12}^{\infty}(t)}{2}\right] \qquad \text{for } |t| < a, \tag{15}$$

in which $H(t) = \dfrac{-1}{\pi \mathrm{i} X(t)}\displaystyle\int_{-a}^{a} \frac{X(s)\sigma_{12}^{\infty}(s)}{s-t}\mathrm{d}s$.

On the other hand, from Eq.(2a) yields

$$\sigma_{11}^+(t) = \mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 Q_k}{S}H(t) - \sigma_{12}^{\infty}(t)\mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 Q_k}{S} \qquad \text{for } |t| < a. \tag{16}$$

Letting $t \to a$, the NST for the Problem B is given by

$$T = -\sigma_{12}^{\infty}(a)\mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 Q_k}{S}. \tag{17}$$

4 Problem C

On the real axis x_1, $\sigma_{22}^{\infty} = 0$, $\sigma_{12}^{\infty} = 0$ and $z_1 = z_2 = z_3 = t$, then from the Eq.(2c) and Eq.(2b) yields

$$\sum_{k=1}^{3}\Phi_k(k) = 0, \ \sum_{k=1}^{3}\mu_k\Phi_k(t) = 0 \qquad \text{for all } t. \tag{18}$$

The boundary values on D_2 in Eq.(2e) become

$$\begin{cases} D_2^+ = \dfrac{S}{S_k}\Phi_k^+(t) + \dfrac{\bar{S}}{\bar{S}_k}\bar{\Phi}_k^-(t), \\ D_2^- = \dfrac{S}{S_k}\Phi_k^-(t) + \dfrac{\bar{S}}{\bar{S}_k}\bar{\Phi}_k^+(t) \qquad \text{for } |t| < a,\ k = 1, 2, 3, \end{cases} \tag{19}$$

in which $S_1 = \mu_3 - \mu_2$; $S_2 = \mu_1 - \mu_3$ and $S_3 = \mu_2 - \mu_1$.

The Rieman-Hilbert formulation is obtained

$$\begin{cases} \left[\dfrac{S}{S_k}\Phi_k(t) + \dfrac{\bar{S}}{\bar{S}_k}\bar{\Phi}_k(t)\right]^+ + \left[\dfrac{S}{S_k}\Phi_k(t) + \dfrac{\bar{S}}{\bar{S}_k}\bar{\Phi}_k(t)\right]^- = 2D_2^{\infty} \qquad \text{for } |t| < a, \\ \left[\dfrac{S}{S_k}\Phi_k(t) - \dfrac{\bar{S}}{\bar{S}_k}\bar{\Phi}_k(t)\right]^+ + \left[\dfrac{S}{S_k}\Phi_k(t) - \dfrac{\bar{S}}{\bar{S}_k}\bar{\Phi}_k(t)\right]^- = 0 \qquad \text{for } |t| < a. \end{cases} \tag{20}$$

The solution can be expressed by

$$\Phi_k(z_k) = \frac{S_k}{S}\frac{-1}{2\pi \mathrm{i} X(z_k)}\int_{-a}^{a}\frac{X^+(t)D_2^{\infty}(t)}{t - z_k}\mathrm{d}t \qquad (k = 1, 2, 3). \tag{21}$$

Using the Plemelj formulation in Eq.(21) and let $z_k \to t^+$, we obtain

$$\Phi_k^+(t) = \frac{S_k}{S}\left[\frac{K(t)}{2} + \frac{D_2^{\infty}}{2}\right] \qquad \text{for } |t| < a, \tag{22}$$

in which $K(t) = \dfrac{1}{\pi \mathrm{i} X(t)}\displaystyle\int_{-a}^{a}\frac{Q(s)D_2^{\infty}(s)}{s-t}\mathrm{d}s$.

On the other hand, from Eq.(2a) yields

$$\sigma_{11}^+(t) = \mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 S_k}{S}K(t) + D_2^{\infty}(t)\mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 S_k}{S} \qquad \text{for } |t| < a. \tag{23}$$

Letting $t \to a$, the NST for the Problem C is given by

$$T = D_2^{\infty}(a)\mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2 S_k}{S}. \tag{24}$$

5 Solution and Discussions

Solution to a piezoelectric crack problem, in which a finite crack of length $2a$ is embeded in an unbounded piezoelectric medium subject to far uniform mechanical loads σ_{12}^{∞}, σ_{22}^{∞} and electrical loads D_2^{∞}, can be given by

$$T = \mathrm{Re}\sum_{k=1}^{3}\frac{\mu_k^2}{S}(D_2^{\infty} S_k - \sigma_{22}^{\infty} P_k - \sigma_{12}^{\infty} Q_k). \tag{25}$$

From above equation, the values of NST depend on

- The influence from far uniform mechanical loads.
- The influence from far uniform electrical loads.
- The influence from the material nature.

If in Eq.(2) we neglect the terms containing the electrical variables, the problem is reduced to one of purely anisotropic elasticity, and the pertinent parameters in Eq.(2) are as follows:

$$\mu_3 = 0,\ \lambda_1 = \lambda_2 = 0,\ \lambda_3 = 1, \tag{26}$$

and

$$\begin{gathered} S = -\mu_1 + \mu_2,\ P_1 = \mu_2,\ P_2 = -\mu_1,\ P_3 = 0, \\ Q_1 = -Q_1 = 1,\ Q_3 = 0. \end{gathered} \tag{27}$$

then Eq.(19) becomes

$$T = \sigma_{22}^{\infty}\mathrm{Re}[\mu_1\mu_2] + \sigma_{12}^{\infty}\mathrm{Re}[\mu_1 + \mu_2]. \tag{28}$$

This equation is the same as the anisotropic case[9].

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